1 Tensor products and direct sums [ @ ] [ @ ]

. They are to combine two Hilbert spaces into One. (vector)

a tenson products

· for a vector if EV and the other is EW

-D BBB C VBW.

ex. a system of two spin-{ pantizles.

the state ket of the system:

 $|\Psi\rangle = C_{\uparrow\uparrow} |\uparrow\rangle_{,} \otimes |\uparrow\rangle_{2} + C_{\uparrow\downarrow} |\uparrow\rangle_{,} \otimes |\downarrow\rangle_{2}$   $+ C_{\downarrow\uparrow} |\downarrow\rangle_{,} \otimes |\uparrow\rangle_{2} + C_{\downarrow\downarrow} |\downarrow\rangle_{,} \otimes |\downarrow\rangle_{2}$ 

· Addition of two operators that are in different H-spacer.

 $= \mathbb{P} \quad \bigcirc_{v} + \bigcirc_{\omega} \equiv \bigcirc_{v} \otimes \mathbb{I}_{\omega} + \mathbb{I}_{v} \otimes \bigcirc_{\omega}$ 

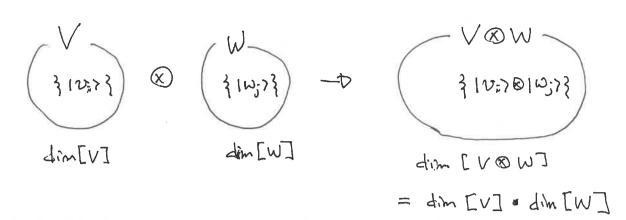
· Operator - (cet multiplication (matrix) (vector)

 $\Rightarrow (5, + 5) | 17 = (+ (5, 17), ) \otimes | 1 \rangle + | 17, \otimes (5, 17)_2) ]$ 

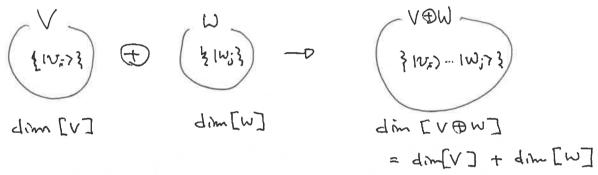
Rule 777 (Or8In+ Iv8Ow). (1278167)

 $= \left( O_{v}(v) \otimes \left( I_{\omega}(\omega) \right) + \left( I_{v}(v) \right) \otimes \left( O_{\omega}(\omega) \right) \right)$ 

a base kets and dimension of the combined space

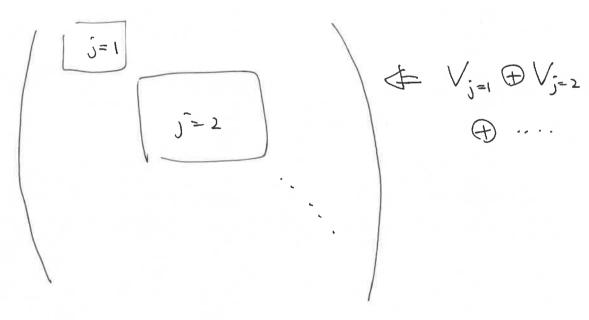


b. Direct sums.



ex natrix representation of D(R)

D(S)



"Block - diagonal".

we have already seen the tensor product many times.

$$\neg \circ | (\alpha, \gamma, 2) = | (\alpha) \otimes | (\gamma) \otimes | (2)$$

$$H = \frac{\vec{p}^2}{2m} = \frac{1}{2m} \left( \left( \vec{p}_z \otimes \vec{I}_z \otimes \vec{I}_z \right) + \left( \vec{I}_z \otimes \hat{p}_z \otimes \vec{I}_z \right) + \left( \vec{I}_z \otimes \vec{I}_z \otimes \hat{p}_z \right)^2 \right)$$

C. notations

1. DO NOT mix then index ordering.

ex. 
$$|\uparrow\rangle$$
,  $\otimes$   $|\downarrow\rangle_2$  +  $|\uparrow\rangle_2$   $\otimes$   $|\downarrow\rangle_1$  ( $\times$ )  $|\uparrow\rangle$ ,  $\otimes$   $|\downarrow\rangle_2$  +  $|\downarrow\rangle$ ,  $\otimes$   $|\uparrow\rangle_2$  ( $\bigcirc$ )

2. (8) are often omitted for brevity.

ex. 
$$|17\rangle_{\cdot}\otimes |11\rangle_{2} + |11\rangle_{\cdot}\otimes |17\rangle_{2} = |17\rangle_{11}\rangle + |11\rangle_{17}\rangle$$
: assuming that index ordering (1,2) in fixed.

2) Simple examples of Angular- Momentum addition

• base ket of a spin - 1 particle in space.
$$|x;1\rangle \equiv |x\rangle \otimes |1\rangle, |x;1\rangle \equiv |x\rangle \otimes |1\rangle$$

· Wave function:

$$\langle x \rangle \uparrow | x \rangle = \psi_{\tau}(x)$$
,  $\langle x \rangle \cup | x \rangle = \psi_{\tau}(x)$ .

or 
$$\Psi(x) \equiv \begin{pmatrix} \psi_{\gamma}(x) \\ \psi_{\psi}(x) \end{pmatrix}$$

· L.S coupling (spin-ordit interaction)

$$\vec{J} = \vec{L} + \vec{S} \quad (\vec{L} \cdot \vec{S} = \vec{J}^2 - \vec{L}^2 - \vec{S}^2)$$

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- base vectors ( bets )

$$|S, M_s\rangle: \left|\frac{1}{2}, \frac{1}{2}\right\rangle \doteq {0 \choose 0}, \left|\frac{1}{2}, \frac{1}{2}\right\rangle \doteq {0 \choose 1}$$

$$|l, m_{e}\rangle: |l, l\rangle \doteq \begin{pmatrix} 0 \\ 0 \end{pmatrix}, |l, 0\rangle \doteq \begin{pmatrix} 0 \\ 0 \end{pmatrix}, |l, -l\rangle \doteq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Angular momentum operators

$$S=\frac{1}{2}$$
:  $S_{+}=\frac{1}{2}$   $S_{+}=\frac{1}{2}$ 

$$k=1: L_{+}=h \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, L_{-}=h \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

in the form of the tensor product (l=1) & (J=\frac{1}{2})

$$\int_{\mathcal{Z}} = \left[ \begin{array}{c} \mathbb{Z} \\ \mathbb$$

eigenfæts.

This obvious that

$$\begin{vmatrix}
3 & 3 \\
2 & 3
\end{vmatrix} = \begin{vmatrix}
0 & m_0 = 1 \\
0 & m_0 = 1
\end{vmatrix}$$

The check of  $J_{+} \begin{vmatrix} 3 & 3 \\ 2 & 2 \end{vmatrix} = 0$ 

The check of  $J_{+} \begin{vmatrix} 3 & 3 \\ 2 & 2 \end{vmatrix} = 0$ 

Lovering with J :

$$J_{-} \left( \frac{3}{2}, \frac{3}{2} \right) = t_{-} \left( \frac{3}{2}, \frac{3}{2} \right)$$

$$= t_{1} \left( \frac{3}{2}, \frac{3}{2} \right)$$

$$= t_{2} \left( \frac{3}{2}, \frac{3}{2} \right)$$

$$= t_{3} \left( \frac{3}{2}, \frac{3}{2} \right)$$

$$= t_{4} \left( \frac{3}{2}, \frac{3}{2} \right)$$

$$= t_{5} \left( \frac{3}{2}, \frac{3}{2} \right)$$

$$= t_{6} \left( \frac{3}{2}, \frac{3}{2} \right)$$

$$= t_{7} \left( \frac{3}{2}, \frac{3}{2} \right)$$

Raising with 
$$J_{+}$$
:

$$J_{+} \left(\frac{3}{2}, -\frac{3}{2}\right) = t_{1} \left(\frac{3}{6}\right) = t_{1} \left(\frac{3}{2}, -\frac{1}{2}\right)$$

From  $\left(\frac{3}{2} - \frac{1}{2}\right) = J_{+}(j, m)$ 

also, by just using operators and leets. J\_ 13,3/ = (L-8I + I8S\_) (11,1) 811/2) = +12 11.07 8 1= +7 + + 11.17 8 1= +13 13,12>  $\Rightarrow \left| \frac{3}{3}, \frac{1}{2} \right\rangle = \sqrt{\frac{3}{3}} \left| 1, 0 \right\rangle \otimes \left| \frac{1}{2}, \frac{1}{2} \right\rangle + \sqrt{\frac{1}{3}} \left| 1, 1 \right\rangle \otimes \left| \frac{1}{2}, -\frac{1}{2} \right\rangle$ = (1/2)  $J_{+}|_{\frac{2}{3}},-\frac{3}{2}\rangle = \left( L_{+} \otimes I + I \otimes S_{+} \right) \left( |1|,-1\rangle \otimes |\frac{1}{2},-\frac{1}{2}\rangle \right)$ = \( \langle \l - (0, 0) by using the orthogonality D'convention", 11、12 = 「」11.07811、12 (13)1.17811、127 = D" Clebsch-Gordan Coefficients"

 $\left| \frac{13}{2}, \frac{1}{2} \right\rangle = \left( \frac{13}{3} \right) \left| \frac{1}{3} \right\rangle \left| \frac{1}{3} \right\rangle$ 

orthogonal matrix

From 
$$\{|l,m_2|\otimes|s,m_5\}\}$$
 to  $\{|j,m\rangle\}$  as

$$|l,m_2|\otimes|s,m_5\}\} = |j,m\rangle$$

$$|l,m_3|\otimes|s,m_5\rangle = |j,m\rangle$$

$$|l,m_2|\otimes|s,m_5\rangle = |j,m\rangle$$

$$|l,m_3|\otimes|s,m_5\rangle = |j,m\rangle$$

Verify: 
$$U J_2 U^{\dagger} = t \begin{pmatrix} \frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

\* How to read out the table of CG coefficients

